

# Itô's formula, the stochastic exponential and change of measure on general time scales.

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## Abstract

We provide an Itô's formula for stochastic dynamical equation on general time scales. Based on this Itô's formula we give a closed form expression for stochastic exponential on general time scales. We then demonstrate a Girsanov's change of measure formula in the case of general time scales.

*Keywords:* Itô's formula, stochastic exponential, change of measure, Girsanov's theorem.

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## 1 Introduction

The theory of dynamical equation on time scales ([3]) has attracted many researches recently. In particular, attempts of extension to stochastic dynamical equations and stochastic analysis on general time scales have been made in several previous works ([5], [4], [10], [8], [7]). In the work [4] the authors mainly work with a discrete time scale; in [5] the authors introduce an extension of a function and define the stochastic as well as deterministic integrals as the usual integrals for the extended function; in [10] the authors make use of their results on the quadratic variation of a Brownian motion ([9]) on time scales and, based on this, they define the stochastic integral via a generalized version of the Itô isometry; in [7] the authors introduce the so called  $\nabla$ -stochastic integral via the backward jump operator and they also derive an Itô's formula based on this definition of the stochastic integral. We notice that different previous works adopt different notions of the stochastic integral and there lacks a uniform and coherent theory of a stochastic calculus on general time scales.

The purpose of the present article is to fill in this gap. We will be mainly working under the framework of [5], in that we define our stochastic integral using the definition

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given in [5]. We then present a general Itô's formula for stochastic dynamical equations under the framework of [5]. Our Itô's formula works for general time scales, and thus fills the gap left in [4], which deals with only discrete time scales. By making use of the Itô's formula we obtain a closed-form expression for the stochastic exponential on general times scales. We will then demonstrate a change of measure (Girsanov's) theorem for stochastic dynamical equation on time scales.

We would like to point out that our change of measure formula is different from the continuous process case in that the density function is not given by the stochastic exponential, but rather is found by the fact that the process on the time scale can be extended to a continuous process simply by linear extension.

It is also worth mentioning that our construction is different from [2] in that we are working with the case that the time parameter of the process is running on a time scale, whereas in [2] and related works (e.g. [11], [1], [6]) the authors are working with the case that the state space of the process is a time scale.

The paper is organized as follows. In Section 2 we discuss some basic set-up for time scales calculus. In Section 3 we will briefly review the results in [5] and define the stochastic integral and stochastic dynamical equation on time scales. In Section 4 we present and prove our Itô's formula. In Section 5 we discuss the formula for stochastic exponential. In Section 6 we prove the change of measure (Girsanov's) formula.

## 2 Set-up: Basics of time scales calculus.

A *time scale*  $\mathbb{T}$  is an arbitrary nonempty closed subset of the real numbers  $\mathbb{R}$ , where we assume that  $\mathbb{T}$  has the topology that it inherits from the real numbers  $\mathbb{R}$  with the standard topology.

We define the *forward jump operator* by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \text{ for all } t \in \mathbb{T} \text{ such that this set is non-empty ,}$$

and the *backward jump operator* by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\} \text{ for all } t \in \mathbb{T} \text{ such that this set is non-empty .}$$

Let  $t \in \mathbb{T}$ . If  $\sigma(t) > t$ , then  $t$  is called *right-scattered*. If  $\sigma(t) = t$ , then  $t$  is called *right dense*. If  $\rho(t) < t$ , then  $t$  is called *left-scattered*. If  $\rho(t) = t$ , then  $t$  is called *left-dense*. Moreover, the sets  $\mathbb{T}^\kappa$  and  $\mathbb{T}_\kappa$  are derived from  $\mathbb{T}$  as follows: If  $\mathbb{T}$  has a left-scattered maximum, then  $\mathbb{T}^\kappa$  is the set  $\mathbb{T}$  without that left-scattered maximum; otherwise,  $\mathbb{T}^\kappa = \mathbb{T}$ . If  $\mathbb{T}$  has a right-scattered minimum, then  $\mathbb{T}_\kappa$  is the set  $\mathbb{T}$  without that right-scattered minimum; otherwise,  $\mathbb{T}_\kappa = \mathbb{T}$ . The *graininess function* is defined by  $\mu(t) = \sigma(t) - t$  for all  $t \in \mathbb{T}^\kappa$ .

Notice that since  $\mathbb{T}$  is closed, for any  $t \in \mathbb{T}$ , the points  $\sigma(t)$  and  $\rho(t)$  are belonging to  $\mathbb{T}$ .

For a set  $A \subset \mathbb{R}$  we denote the set  $A_{\mathbb{T}} = A \cap \mathbb{T}$ .

### 3 Stochastic integrals and stochastic differential equations on time scales.

We will adopt the definitions introduced in [5] as our definition of a Brownian motion and Itô's stochastic integral on time scales. In the next section we will derive an Itô's formula corresponding to the stochastic integral defined in such a way.

**Definition 1.** *A Brownian motion indexed by a time scale  $\mathbb{T}$  is an adapted stochastic process  $\{W_t\}_{t \in \mathbb{T}}$  on a filtered probability space  $(\Omega, \mathcal{F}_t, \mathbf{P})$  such that*

- (1)  $\mathbf{P}(W_0 = 0) = 1$ ;
- (2) *If  $0 \leq s < t \leq 1$  and  $s, t \in \mathbb{T}$  then the increment  $W_t - W_s$  is independent of  $\mathcal{F}_s$  and is normally distributed with mean 0 and variance  $t - s$ ;*
- (3) *The process  $W_t$  is almost surely continuous on  $\mathbb{T}$ .*

Note that the property (3) is proved in the work [8].

For a random function  $f : [0, \infty)_{\mathbb{T}} \times \Omega \rightarrow \mathbb{R}$  we define the *extension*  $\tilde{f} : [0, \infty) \times \Omega \rightarrow \mathbb{R}$  by

$$\tilde{f}(t, \omega) = f(\sup[0, t]_{\mathbb{T}}, \omega) \tag{1}$$

for all  $t \in [0, \infty)$ .

We shall make use of the definitions given in [5] for the classical Lebesgue and Riemann integral. For any random function  $f : [0, \infty)_{\mathbb{T}} \times \Omega \rightarrow \mathbb{R}$  and  $T < \infty$  we define its  $\Delta$ -Riemann (Lebesgue) integral as

$$\int_0^T f(t, \omega) \Delta t = \int_0^T \tilde{f}(t, \omega) dt ,$$

where the integral on the right hand side of the above equation is interpreted as a standard Riemann (Lebesgue) integral. In a similar way, the work [5] defines a stochastic integral for an  $L^2([0, T]_{\mathbb{T}})$ -progressively measurable random function  $f(t, \omega)$  as

$$\int_0^T f(t, \omega) \Delta W_t = \int_0^T \tilde{f}(t, \omega) dW_t ,$$

where again the right hand side of the above equation is interpreted as a standard Itô's stochastic integral. Note that the way (1) that we define the extension guarantees that the function  $\tilde{f}(t, \omega)$  is progressively measurable.

In [5] the authors then defined the solution of the  $\Delta$ -stochastic differential equation indicated by the notation

$$\Delta X_t = b(t, X)\Delta t + \sigma(t, X)\Delta W_t, \quad (2)$$

as the process  $\{X_t\}_{t \in [0, T]_{\mathbb{T}}}$  such that

$$X_{t_2} - X_{t_1} = \int_{t_1}^{t_2} b(t, X_t)\Delta t + \int_{t_1}^{t_2} \sigma(t, X_t)\Delta W_t,$$

with the deterministic and stochastic integrals on the right-hand side of the above equality interpreted as was just mentioned. Under the condition of continuity in the  $t$ -variable and uniform Lipschitz continuity in the  $x$ -variable of the functions  $b(t, x)$  and  $\sigma(t, x)$ , together with being no worse than linear growth in  $x$ -variable, existence and pathwise uniqueness of strong solution to (2) are proved in [5].

## 4 Itô's formula for stochastic integrals on time scales.

We will make use of the following fact that is simple to prove.

**Proposition 1.** *The set of all left-scattered or right-scattered points of  $\mathbb{T}$  is at most countable.*

**Proof.** If  $x \in \mathbb{T}$  is a right-scattered point, then  $I_x = (x, \sigma(x))$  is an open interval such that  $I_x \cap \mathbb{T} = \emptyset$ . Similarly, if  $x \in \mathbb{T}$  is a left-scattered point, then  $I_x = (\rho(x), x)$  is an open interval such that  $I_x \cap \mathbb{T} = \emptyset$ . Suppose  $x < y$  and  $x, y \in \mathbb{T}$ . We then distinguish four different cases.

Case 1: both  $x$  and  $y$  are right-scattered. We argue that in this case we have  $I_x \cap I_y = \emptyset$ . Suppose this is not the case, then we must have  $\sigma(x) > y$ . But we see that  $\sigma(x) = \inf\{s > x : s \in \mathbb{T}\}$  and  $y \in \mathbb{T}$ . So we must have  $\sigma(x) \leq y$ . We arrive at a contradiction;

Case 2: both  $x$  and  $y$  are left-scattered. This case is similar to Case 1 and we conclude that  $I_x \cap I_y = \emptyset$ ;

Case 3:  $x$  is left-scattered,  $y$  is right-scattered. In this case we see that  $I_x = (\rho(x), x)$  and  $I_y = (y, \sigma(y))$ , as well as  $x < y$ . This implies that  $I_x \cap I_y = \emptyset$ ;

Case 4:  $x$  is right-scattered,  $y$  is left-scattered. In this case  $I_x = (x, \sigma(x))$  and  $I_y = (\rho(y), y)$ . If  $\sigma(x) \leq \rho(y)$ , then  $I_x \cap I_y = \emptyset$ . If  $\sigma(x) > \rho(y)$ , then we see that  $(x, y) = I_x \cup I_y$  so that  $(x, y) \cap \mathbb{T} = \emptyset$ . That implies further  $\sigma(x) = y$  and  $\rho(y) = x$ , i.e.,  $I_x = I_y$ .

Thus we see that for all points  $x \in \mathbb{T}$  being left or right-scattered, the set of all open intervals of the form  $I_x$  are disjoint subsets of  $\mathbb{R}$ . Henceforth there are at most countably many such intervals. Each such interval corresponds to one or two endpoints in  $\mathbb{T}$  that are either left or right-scattered. Thus the total number of left or right-scattered points in  $\mathbb{T}$  are at most countably many.  $\square$

Let  $C$  be the (at most) countable set of all left-scattered or right-scattered points of  $\mathbb{T}$ . As we have already seen in the proof of the previous Proposition, the set  $C$  corresponds to at most countably many open intervals  $\mathcal{I} = \{I_1, I_2, \dots\}$  such that (1) for any  $k \neq l$ ,  $I_k \cap I_l = \emptyset$ ; (2) either the left endpoint or right endpoint or both endpoints of any of the  $I_k$ 's are in  $\mathbb{T}$ , and are left or right-scattered; (3)  $I_k \cap \mathbb{T} = \emptyset$  for any  $k = 1, 2, \dots$ ; (4) any point in  $C$  is a left or right-endpoint of one of the  $I_k$ 's.

We will denote  $I_k = (s_{I_k}^-, s_{I_k}^+)$ . Since for any  $x \in \mathbb{T}$ , the points  $\sigma(x)$  and  $\rho(x)$  are in  $\mathbb{T}$ , we further infer that for any such interval  $I_k$ , we have  $s_{I_k}^-$  and  $s_{I_k}^+$  are in  $\mathbb{T}$ , so that  $s_{I_k}^-$  is right-scattered and  $s_{I_k}^+$  is left-scattered.

We then establish the following Itô's formula.

For any two points  $t_1, t_2 \in \mathbb{T}$ ,  $t_1 \leq t_2$ , and any open interval  $I_k \in \mathcal{I}$ , such that  $I_k \cap [t_1, t_2] \neq \emptyset$ , we have  $I_k \subset (t_1, t_2)$ . This is because if not the case, then  $t_1$  or  $t_2$  will belong to  $I_k$ , contradictory to the fact that  $I_k \cap \mathbb{T} = \emptyset$ . We conclude that

$$\{I_k \in \mathcal{I} : I_k \cap [t_1, t_2] \neq \emptyset\} = \{I_k \in \mathcal{I} : I_k \subset (t_1, t_2)\}.$$

**Theorem 1.** (Itô's formula) *For any function  $f \in \mathbf{C}^{(2)}(\mathbb{R}_+, \mathbb{R})$  and any  $t_1 \leq t_2$ ,  $t_1, t_2 \in [0, \infty)_{\mathbb{T}}$  we have*

$$\begin{aligned} & f(t_2, W_{t_2}) - f(t_1, W_{t_1}) \\ = & \int_{t_1}^{t_2} \frac{\partial f}{\partial t}(s, W_s) \Delta s + \int_{t_1}^{t_2} \frac{\partial f}{\partial x}(s, W_s) \Delta W_s + \frac{1}{2} \int_{t_1}^{t_2} \frac{\partial^2 f}{\partial x^2}(s, W_s) \Delta s \\ & + \sum_{I_k \in \mathcal{I}, I_k \subset (t_1, t_2)} \left[ f(s_{I_k}^+, W_{s_{I_k}^+}) - f(s_{I_k}^-, W_{s_{I_k}^-}) - \frac{\partial f}{\partial t}(s_{I_k}^-, W_{s_{I_k}^-})(s_{I_k}^+ - s_{I_k}^-) \right. \\ & \quad \left. - \frac{\partial f}{\partial x}(s_{I_k}^-, W_{s_{I_k}^-})(W_{s_{I_k}^+} - W_{s_{I_k}^-}) - \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(s_{I_k}^-, W_{s_{I_k}^-})(s_{I_k}^+ - s_{I_k}^-) \right]. \end{aligned} \quad (3)$$

**Proof.** We will make use of the following version (Peano form) of Taylor's theorem: for any function  $f \in \mathbf{C}^{(2)}(\mathbb{R}_+, \mathbb{R})$  and any  $s_1, s_2, x_1, x_2$  we have

$$f(s_2, x_2) - f(s_1, x_1) = \frac{\partial f}{\partial t}(s_1, x_1)(s_2 - s_1) + \frac{\partial f}{\partial x}(s_1, x_1)(x_2 - x_1) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(s_1, x_1)(x_2 - x_1)^2 + R(s_1, s_2; x_1, x_2), \quad (4)$$

where

$$|R(s_1, s_2; x_1, x_2)| \leq r(|s_2 - s_1|)|s_2 - s_1| + r(|x_2 - x_1|)(x_2 - x_1)^2 ,$$

and  $r : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is an increasing function with  $\lim_{u \downarrow 0} r(u) = 0$ .

Consider a partition  $\pi^{(n)}$ :  $t_1 = s_0 < s_1 < \dots < s_n = t_2$ , such that (1) each  $s_i \in \mathbb{T}$ ; (2)  $\max_i (\rho(s_i) - s_{i-1}) \leq \frac{1}{2^n}$  for  $i = 1, 2, \dots, n$ . Notice that by definition  $\rho(s_i) = \sup\{s < s_i : s \in \mathbb{T}\}$ , so that we can always find  $s_{i-1} \in \mathbb{T}$  so that  $\rho(s_i) - s_{i-1}$  is sufficiently small.

Let the sets  $C$  and  $\mathcal{I}$  be defined as before. Let us fix a partition  $\pi^{(n)}$ , and consider a classification of its corresponding intervals  $(s_{i-1}, s_i)$ ,  $i = 1, 2, \dots, n$ . We will classify all intervals  $(s_{i-1}, s_i)$  such that for all  $I_k \in \mathcal{I}$  we have  $I_k \cap (s_{i-1}, s_i) = \emptyset$  as class (a); and we classify all intervals  $(s_{i-1}, s_i)$  such that there exist some  $I_k \in \mathcal{I}$  with  $(s_{i-1}, s_i) \cap I_k \neq \emptyset$  as class (b). For an interval  $(s_{i-1}, s_i)$  in class (a), since for all  $I_k \in \mathcal{I}$  we have  $I_k \cap (s_{i-1}, s_i) = \emptyset$ , we see that  $\rho(s_i) = s_i$ . Because otherwise  $(\rho(s_i), s_i)$  will be one of the  $I_k$ 's. Thus in this case we have  $s_i - s_{i-1} < \frac{1}{2^n}$ . For an interval  $(s_{i-1}, s_i)$  in class (b), since both  $s_{i-1}$  and  $s_i$  are in  $\mathbb{T}$ , we see that we have in fact  $I_k \subseteq (s_{i-1}, s_i)$ . In this case either  $I_k = (s_{i-1}, s_i)$ , or  $I_k \neq (s_{i-1}, s_i)$ . If the latter happens, then  $(\rho(s_i), s_i) \in \mathcal{I}$  is one of the  $I_k$ 's and  $\rho(s_i) - s_{i-1} < \frac{1}{2^n}$ . We also see from the above analysis that all  $I_k$ 's are contained in intervals  $(s_{i-1}, s_i)$  that belong to class (b). On the other hand, each interval  $(s_{i-1}, s_i)$  is either entirely one of the  $I_k$ 's, or it contains an interval  $(\rho(s_i), s_i)$  that is one of the  $I_k$ 's. For the latter case, i.e., when  $s_{i-1} < \rho(s_i) < s_i$ , the set of intervals of the form  $(s_{i-1}, \rho(s_i))$  are disjoint open intervals such that

$$\sum_{(s_{i-1}, s_i) \in (b), s_{i-1} < \rho(s_i) < s_i} (\rho(s_i) - s_{i-1}) < \frac{n}{2^n} . \quad (5)$$

Now we have

$$\begin{aligned} & f(t_2, W_{t_2}) - f(t_1, W_{t_1}) \\ &= \sum_{i=1}^n [f(s_i, W_{s_i}) - f(s_{i-1}, W_{s_{i-1}})] \\ &= \sum_{(a)} [f(s_i, W_{s_i}) - f(s_{i-1}, W_{s_{i-1}})] + \sum_{(b)} [f(s_i, W_{s_i}) - f(s_{i-1}, W_{s_{i-1}})] \\ &= (I) + (II) . \end{aligned} \quad (6)$$

We apply (4) term by term in part (I) of (6), and we get

$$\begin{aligned}
& (I) \\
&= \sum_{(a)} [f(s_i, W_{s_i}) - f(s_{i-1}, W_{s_{i-1}})] \\
&= \sum_{(a)} \left[ \frac{\partial f}{\partial t}(s_{i-1}, W_{s_{i-1}})(s_i - s_{i-1}) + \frac{\partial f}{\partial x}(s_{i-1}, W_{s_{i-1}})(W_{s_i} - W_{s_{i-1}}) \right. \\
&\quad \left. + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(s_{i-1}, W_{s_{i-1}})(W_{s_i} - W_{s_{i-1}})^2 + R(s_{i-1}, s_i; W_{s_{i-1}}, W_{s_i}) \right] \\
&= \sum_{i=1}^n \left[ \frac{\partial f}{\partial t}(s_{i-1}, W_{s_{i-1}})(s_i - s_{i-1}) + \frac{\partial f}{\partial x}(s_{i-1}, W_{s_{i-1}})(W_{s_i} - W_{s_{i-1}}) \right] \\
&\quad + \left( \sum_{(a)} \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(s_{i-1}, W_{s_{i-1}})(W_{s_i} - W_{s_{i-1}})^2 + \sum_{(b)} \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(s_{i-1}, W_{s_{i-1}})(s_i - s_{i-1}) \right) \\
&\quad - \sum_{(b)} \left[ \frac{\partial f}{\partial t}(s_{i-1}, W_{s_{i-1}})(s_i - s_{i-1}) + \frac{\partial f}{\partial x}(s_{i-1}, W_{s_{i-1}})(W_{s_i} - W_{s_{i-1}}) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(s_{i-1}, W_{s_{i-1}})(s_i - s_{i-1}) \right] \\
&\quad + \sum_{(a)} R(s_{i-1}, s_i; W_{s_{i-1}}, W_{s_i}) . \\
&= (III)_1 + (III)_2 + (IV) + (V) .
\end{aligned} \tag{7}$$

We have the following four convergence results:

*Convergence Result 1.1.* By Lemma 1 (12) and (13) established below we have

$$\mathbf{P} \left( (III)_1 \rightarrow \int_{t_1}^{t_2} \frac{\partial f}{\partial t}(s, W_s) \Delta s + \int_{t_1}^{t_2} \frac{\partial f}{\partial x}(s, W_s) \Delta W_s \text{ as } n \rightarrow \infty \right) = 1 . \tag{8}$$

*Convergence Result 1.2.* By Lemma 2 (14) and Lemma 1 (12) established below we have

$$\mathbf{P} \left( (III)_2 \rightarrow \int_{t_1}^{t_2} \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(s, W_s) \Delta s \text{ as } n \rightarrow \infty \right) = 1 . \tag{9}$$

*Convergence Result 2.* We have, with probability 1, that

$$(V) = \sum_{(a)} R(s_{i-1}, s_i; W_{s_{i-1}}, W_{s_i}) \rightarrow 0$$

as  $n \rightarrow \infty$ .

In fact, by the Kolmogorov–Čentsov theorem proved in Theorem 3.1 of [8] we know that for almost all trajectory of  $W_t$  on  $\mathbb{T}$ , for each fixed trajectory  $W_t(\omega)$ , there exist an  $n_0 = n_0(\omega)$  such that for all  $n \geq n_0$ , for a partition  $\pi^{(n)}$  with a classification of its intervals  $(s_{i-1}, s_i)$  into classes (a) and (b) as above,  $\sup_{(a)} |W_{s_i} - W_{s_{i-1}}| \leq \frac{\delta}{2\gamma n/5}$  for some fixed  $\delta > 0$  and  $\gamma > 0$ . From here we can estimate

$$\begin{aligned}
& \mathbf{E} \sum_{(a)} R(s_{i-1}, s_i; W_{s_{i-1}}, W_{s_i}) \\
& \leq \mathbf{E} \sum_{(a)} [r(s_i - s_{i-1})(s_i - s_{i-1}) + r(|W_{s_i} - W_{s_{i-1}}|)(W_{s_i} - W_{s_{i-1}})^2] \\
& \leq r \left( \frac{1}{2^n} \right) (t_2 - t_1) + r \left( \frac{\delta}{2^{\gamma n/5}} \right) (t_2 - t_1) ,
\end{aligned}$$

i.e.,

$$\mathbf{P} \left( \lim_{n \rightarrow \infty} \sum_{(a)} R(s_{i-1}, s_i; W_{s_{i-1}}, W_{s_i}) = 0 \right) = 1 . \quad (10)$$

*Convergence Result 3.* Let

$$\begin{aligned}
(II) + (IV) = A_n = A_n(\omega) &= \sum_{(b)} \left[ f(s_i, W_{s_i}) - f(s_{i-1}, W_{s_{i-1}}) - \frac{\partial f}{\partial t}(s_{i-1}, W_{s_{i-1}})(s_i - s_{i-1}) \right. \\
&\quad \left. - \frac{\partial f}{\partial x}(s_{i-1}, W_{s_{i-1}})(W_{s_i} - W_{s_{i-1}}) - \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(s_{i-1}, W_{s_{i-1}})(s_i - s_{i-1}) \right] ,
\end{aligned}$$

and

$$\begin{aligned}
B_n = B_n(\omega) &= \sum_{I_k \in \mathcal{I}, I_k \subset (t_1, t_2)} \left[ f(s_{I_k}^+, W_{s_{I_k}^+}) - f(s_{I_k}^-, W_{s_{I_k}^-}) - \frac{\partial f}{\partial t}(s_{I_k}^-, W_{s_{I_k}^-})(s_{I_k}^+ - s_{I_k}^-) \right. \\
&\quad \left. - \frac{\partial f}{\partial x}(s_{I_k}^-, W_{s_{I_k}^-})(W_{s_{I_k}^+} - W_{s_{I_k}^-}) - \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(s_{I_k}^-, W_{s_{I_k}^-})(s_{I_k}^+ - s_{I_k}^-) \right] .
\end{aligned}$$

We claim that we have

$$\mathbf{P}(|A_n(\omega) - B_n(\omega)| \rightarrow 0 \text{ as } n \rightarrow \infty) = 1 . \quad (11)$$

In fact, from the analysis that leads to the estimate (5) we see that we can write  $A_n - B_n$  as



$$\begin{aligned}
& A_n - B_n \\
= & \sum_{(s_{i-1}, s_i) \in (b), s_{i-1} < \rho(s_i) < s_i} \left[ f(s_i, W_{s_i}) - f(s_{i-1}, W_{s_{i-1}}) - \frac{\partial f}{\partial t}(s_{i-1}, W_{s_{i-1}})(s_i - s_{i-1}) \right. \\
& \left. - \frac{\partial f}{\partial x}(s_{i-1}, W_{s_{i-1}})(W_{s_i} - W_{s_{i-1}}) - \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(s_{i-1}, W_{s_{i-1}})(s_i - s_{i-1}) \right] \\
& - \sum_{(s_{i-1}, s_i) \in (b), s_{i-1} < \rho(s_i) < s_i} \left[ f(s_i, W_{s_i}) - f(\rho(s_i), W_{\rho(s_i)}) - \frac{\partial f}{\partial t}(\rho(s_i), W_{\rho(s_i)})(s_i - \rho(s_i)) \right. \\
& \left. - \frac{\partial f}{\partial x}(\rho(s_i), W_{\rho(s_i)})(W_{s_i} - W_{\rho(s_i)}) - \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(\rho(s_i), W_{\rho(s_i)})(s_i - \rho(s_i)) \right] \\
& - \sum_{I_k \in \mathcal{I}, I_k \subset (s_{i-1}, \rho(s_i)) \text{ for some } (s_{i-1}, s_i) \in (b)} \left[ f(s_{I_k}^+, W_{s_{I_k}^+}) - f(s_{I_k}^-, W_{s_{I_k}^-}) \right. \\
& \left. - \frac{\partial f}{\partial t}(s_{I_k}^-, W_{s_{I_k}^-})(s_{I_k}^+ - s_{I_k}^-) - \frac{\partial f}{\partial x}(s_{I_k}^-, W_{s_{I_k}^-})(W_{s_{I_k}^+} - W_{s_{I_k}^-}) - \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(s_{I_k}^-, W_{s_{I_k}^-})(s_{I_k}^+ - s_{I_k}^-) \right] \\
= & (VI)_1 + (VI)_2 + (VI)_3 + (VI)_4 - (VII) .
\end{aligned}$$

Here

$$\begin{aligned}
(VI)_1 &= \sum_{(s_{i-1}, s_i) \in (b), s_{i-1} < \rho(s_i) < s_i} [f(\rho(s_i), W_{\rho(s_i)}) - f(s_{i-1}, W_{s_{i-1}})] , \\
(VI)_2 &= \sum_{(s_{i-1}, s_i) \in (b), s_{i-1} < \rho(s_i) < s_i} \left[ \frac{\partial f}{\partial t}(\rho(s_i), W_{\rho(s_i)})(s_i - \rho(s_i)) - \frac{\partial f}{\partial t}(s_{i-1}, W_{s_{i-1}})(s_i - s_{i-1}) \right] \\
&= \sum_{(s_{i-1}, s_i) \in (b), s_{i-1} < \rho(s_i) < s_i} \left[ \left( \frac{\partial f}{\partial t}(\rho(s_i), W_{\rho(s_i)}) - \frac{\partial f}{\partial t}(s_{i-1}, W_{s_{i-1}}) \right) (s_i - s_{i-1}) \right. \\
&\quad \left. - \frac{\partial f}{\partial t}(\rho(s_i), W_{\rho(s_i)})(\rho(s_i) - s_{i-1}) \right] , \\
(VI)_3 &= \sum_{(s_{i-1}, s_i) \in (b), s_{i-1} < \rho(s_i) < s_i} \left[ \frac{\partial f}{\partial t}(\rho(s_i), W_{\rho(s_i)})(W_{s_i} - W_{\rho(s_i)}) - \frac{\partial f}{\partial t}(s_{i-1}, W_{s_{i-1}})(W_{s_i} - W_{s_{i-1}}) \right] \\
&= \sum_{(s_{i-1}, s_i) \in (b), s_{i-1} < \rho(s_i) < s_i} \left[ \left( \frac{\partial f}{\partial t}(\rho(s_i), W_{\rho(s_i)}) - \frac{\partial f}{\partial t}(s_{i-1}, W_{s_{i-1}}) \right) (W_{s_i} - W_{s_{i-1}}) \right. \\
&\quad \left. - \frac{\partial f}{\partial t}(\rho(s_i), W_{\rho(s_i)})(W_{\rho(s_i)} - W_{s_{i-1}}) \right] , \\
(VI)_4 &= \sum_{(s_{i-1}, s_i) \in (b), s_{i-1} < \rho(s_i) < s_i} \left[ \frac{\partial f}{\partial t}(\rho(s_i), W_{\rho(s_i)})(s_i - \rho(s_i)) - \frac{\partial f}{\partial t}(s_{i-1}, W_{s_{i-1}})(s_i - s_{i-1}) \right] \\
&= \sum_{(s_{i-1}, s_i) \in (b), s_{i-1} < \rho(s_i) < s_i} \left[ \frac{\partial f}{\partial t}(s_{i-1}, W_{s_{i-1}})(s_{i-1} - \rho(s_i)) \right. \\
&\quad \left. + \left( \frac{\partial f}{\partial t}(\rho(s_i), W_{\rho(s_i)}) - \frac{\partial f}{\partial t}(s_{i-1}, W_{s_{i-1}}) \right) (s_i - \rho(s_i)) \right] ,
\end{aligned}$$

(VII)

$$= \sum_{I_k \in \mathcal{I}, I_k \subset (s_{i-1}, \rho(s_i)) \text{ for some } (s_{i-1}, s_i) \in (b)} \left[ f(s_{I_k}^+, W_{s_{I_k}^+}) - f(s_{I_k}^-, W_{s_{I_k}^-}) - \frac{\partial f}{\partial t}(s_{I_k}^-, W_{s_{I_k}^-})(s_{I_k}^+ - s_{I_k}^-) - \frac{\partial f}{\partial x}(s_{I_k}^-, W_{s_{I_k}^-})(W_{s_{I_k}^+} - W_{s_{I_k}^-}) - \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(s_{I_k}^-, W_{s_{I_k}^-})(s_{I_k}^+ - s_{I_k}^-) \right].$$

From (5), the Kolmogorov–Čentsov theorem proved in Theorem 3.1 of [8], as well as the fact that  $f \in \mathbf{C}^{(2)}(\mathbb{R}_+, \mathbb{R})$ , we immediately see the claim (11).

Combining the convergence results (8), (9), (10), (11), together with (6) and (7), we establish (3).  $\square$

**Lemma 1.** (Convergence of  $\Delta$ –deterministic and stochastic intergarals)

Given a time scale  $\mathbb{T}$  and  $t_1, t_2 \in \mathbb{T}$ ,  $t_1 < t_2$ ; a probability space  $(\Omega, \mathcal{F}, \mathbf{P})$ ; a Brownian motion  $\{W_t\}_{t \in \mathbb{T}}$  on the time scale  $\mathbb{T}$ , for any progressively measurable random function  $f \in \mathbf{C}([t_1, t_2])$ , viewed as a  $L^2([t_1, t_2]_{\mathbb{T}})$ –progressively measurable random function  $f(t, \omega)$  on  $\mathbb{T}$ , and the families of partitions  $\pi^{(n)} : t_1 = s_0 < s_1 < \dots < s_n = t_2$ ,  $s_0, s_1, \dots, s_n \in \mathbb{T}$ ,  $\max_{i=1,2,\dots,n} (\rho(s_i) - s_{i-1}) < \frac{1}{2^n}$ , we have

$$\mathbf{P} \left( \lim_{n \rightarrow \infty} \sum_{i=1}^n f(s_{i-1}, \omega)(s_i - s_{i-1}) = \int_{t_1}^{t_2} f(s, \omega) \Delta s \right) = 1, \quad (12)$$

$$\mathbf{P} \left( \lim_{n \rightarrow \infty} \sum_{i=1}^n f(s_{i-1}, \omega)(W_{s_i} - W_{s_{i-1}}) = \int_{t_1}^{t_2} f(s, \omega) \Delta W_s \right) = 1. \quad (13)$$

**Proof.** As we have seen in the proof of the Itô’s formula, for a given partition  $\pi^{(n)} : t_1 = s_0 < s_1 < \dots < s_n = t_2$ , such that  $s_i \in \mathbb{T}$  for  $i = 0, 1, \dots, n$ , and  $\max_{i=1,2,\dots,n} (\rho(s_i) - s_{i-1}) < \frac{1}{2^n}$ , we can classify all intervals of the form  $(s_{i-1}, s_i)$  into two classes (a) and (b): class (a) are those open intervals  $(s_{i-1}, s_i)$  such that it does not contain any open intervals  $I_k \in \mathcal{I}$ ; class (b) are those open intervals  $(s_{i-1}, s_i)$  such that it contains at least one open interval  $I_k \in \mathcal{I}$ , the latter of which has endpoints that are left or right scattered.

Let us form a family of partitions  $\sigma^{(n)} : t_1 = r_0 < r_1 < \dots < r_m = t_2$ , so that the partition  $\sigma^{(n)}$  is the partition  $\pi^{(n)}$  together with all points in  $\mathbb{T}$  that are of the form  $r_j = \rho(s_i)$  for some  $s_i$  in the partition  $\pi^{(n)}$ . Note that under this construction we have  $r_0, r_1, \dots, r_m \in \mathbb{T}$ . In fact, for any interval  $(s_{i-1}, s_i)$  in (a), there is an identical interval  $(r_{j-1}, r_j)$  in the partition  $\sigma^{(n)}$  corresponding to it; for any interval  $(s_{i-1}, s_i)$  in (b), there are two intervals  $(r_{j-2}, r_{j-1})$  and  $(r_{j-1}, r_j)$  corresponding to it, so that  $r_{j-1} = \rho(s_i)$ .

And by (5) we know that

$$\sum_{(s_{i-1}, s_i) \in (b), (r_{j-2}, r_{j-1}) \text{ is corresponding to it}} (r_{j-1} - r_{j-2}) < \frac{n}{2^n}.$$

Note that the number  $m$  depends on  $n$  and the partition  $\pi^{(n)}$ :  $m = m(n, \pi^{(n)})$ . In particular  $m \rightarrow \infty$  as  $n \rightarrow \infty$ . For simplicity we will suppress this dependence later in our proof.

Let us recall the definition of deterministic and stochastic  $\Delta$ -integrals as defined in Section 2. Let  $\tilde{f}$  be the extension of  $f$  that we have in (1): for any  $t \in \mathbb{T}$ ,

$$\tilde{f}(t, \omega) = f(\sup[0, t]_{\mathbb{T}}, \omega).$$

Note that if  $t \in \mathbb{T}$  is such that  $\rho(t) = t$ , then  $\tilde{f}(t, \omega) = f(t, \omega)$ , otherwise if  $t \in \mathbb{T}$  is such that  $\rho(t) < t$ , then  $\tilde{f}(t, \omega) = f(\rho(t), \omega)$ . Thus we see that

$$\begin{aligned} \mathbf{P} \left( \lim_{n \rightarrow \infty} \sum_{j=1}^m f(r_{j-1}, \omega)(r_j - r_{j-1}) = \int_{t_1}^{t_2} \tilde{f}(s, \omega) ds \right) &= 1, \\ \mathbf{P} \left( \lim_{n \rightarrow \infty} \sum_{j=1}^m f(r_{j-1}, \omega)(W_{r_j} - W_{r_{j-1}}) = \int_{t_1}^{t_2} \tilde{f}(s, \omega) dW_s \right) &= 1. \end{aligned}$$

So it suffices to prove that

$$\mathbf{P} \left( \lim_{n \rightarrow \infty} \left[ \sum_{i=1}^n f(s_{i-1}, \omega)(s_i - s_{i-1}) - \sum_{j=1}^m f(r_{j-1}, \omega)(r_j - r_{j-1}) \right] = 0 \right) = 1$$

and

$$\mathbf{P} \left( \lim_{n \rightarrow \infty} \left[ \sum_{i=1}^n f(s_{i-1}, \omega)(W_{s_i} - W_{s_{i-1}}) - \sum_{j=1}^m f(r_{j-1}, \omega)(W_{r_j} - W_{r_{j-1}}) \right] = 0 \right) = 1.$$

In fact, for any interval  $(s_{i-1}, s_i)$  in class (a), there exist an interval  $(r_{j-1}, r_j)$  identical to the interval  $(s_{i-1}, s_i)$ , so that

$$f(s_{i-1}, \omega)(s_i - s_{i-1}) - f(r_{j-1}, \omega)(r_j - r_{j-1}) = 0$$

and

$$f(s_{i-1}, \omega)(W_{s_i} - W_{s_{i-1}}) - f(r_{j-1}, \omega)(W_{r_j} - W_{r_{j-1}}) = 0.$$

For any open interval  $(s_{i-1}, s_i)$  in class (b), there are two corresponding intervals  $(r_{j-2}, r_{j-1})$  and  $(r_{j-1}, r_j)$  such that  $r_{j-2} = s_{i-1}$ ,  $r_{j-1} = \rho(s_i)$  and  $r_j = s_i$ . In this case

$$\begin{aligned}
& f(s_{i-1}, \omega)(s_i - s_{i-1}) - f(r_{j-1}, \omega)(r_j - r_{j-1}) - f(r_{j-2}, \omega)(r_{j-1} - r_{j-2}) \\
&= f(s_{i-1}, \omega)(s_i - s_{i-1}) - f(\rho(s_i), \omega)(s_i - \rho(s_i)) - f(s_{i-1}, \omega)(\rho(s_i) - s_{i-1}) \\
&= (f(s_{i-1}, \omega) - f(\rho(s_i), \omega))(s_i - \rho(s_i))
\end{aligned}$$

and

$$\begin{aligned}
& f(s_{i-1}, \omega)(W_{s_i} - W_{s_{i-1}}) - f(r_{j-1}, \omega)(W_{r_j} - W_{r_{j-1}}) - f(r_{j-2}, \omega)(W_{r_{j-1}} - W_{r_{j-2}}) \\
&= f(s_{i-1}, \omega)(W_{s_i} - W_{s_{i-1}}) - f(\rho(s_i), \omega)(W_{s_i} - W_{\rho(s_i)}) - f(s_{i-1}, \omega)(W_{\rho(s_i)} - W_{s_{i-1}}) \\
&= (f(s_{i-1}, \omega) - f(\rho(s_i), \omega))(W_{s_i} - W_{\rho(s_i)}) .
\end{aligned}$$

From the above calculations and the fact that we have (5) and  $f \in \mathbf{C}([t_1, t_2])$ , we see the claim follows.  $\square$

**Lemma 2.** *Given a time scale  $\mathbb{T}$  and  $t_1, t_2 \in \mathbb{T}$ ,  $t_1 < t_2$ ; a probability space  $(\Omega, \mathcal{F}, \mathbf{P})$ ; a Brownian motion  $\{W_t\}_{t \in \mathbb{T}}$  on the time scale  $\mathbb{T}$ . Let any  $L^2([t_1, t_2]_{\mathbb{T}})$ -progressively measurable random function  $f(t, \omega)$  on  $\mathbb{T}$  be defined such that  $\mathbf{E}f^2(t, \omega)$  is uniformly bounded on  $[t_1, t_2]$ . Consider the families of partitions  $\pi^{(n)} : t_1 = s_0 < s_1 < \dots < s_n = t_2$ ,  $s_0, s_1, \dots, s_n \in \mathbb{T}$ ,  $\max_{i=1,2,\dots,n} (\rho(s_i) - s_{i-1}) < \frac{1}{2^n}$ . We classify all the intervals  $(s_{i-1}, s_i)$ ,  $i = 1, 2, \dots, n$  into two classes (a) and (b) as before. Then we have*

$$\mathbf{P} \left( \lim_{n \rightarrow \infty} \left[ \sum_{(a)} f(s_{i-1}, \omega)(W_{s_i} - W_{s_{i-1}})^2 - \sum_{(a)} f(s_{i-1}, \omega)(s_i - s_{i-1}) \right] = 0 \right) = 1 . \quad (14)$$

**Proof.** We notice that for all intervals  $(s_{i-1}, s_i) \in (a)$  we have  $\rho(s_i) = s_{i-1}$  and thus  $s_i - s_{i-1} < \frac{1}{2^n}$ . Let us denote that

$$V_n = \left[ \sum_{(a)} f(s_{i-1}, \omega)(W_{s_i} - W_{s_{i-1}})^2 - \sum_{(a)} f(s_{i-1}, \omega)(s_i - s_{i-1}) \right] .$$

Since  $f(t, \omega)$  is progressively measurable, we see that  $f(s_{i-1}, \omega)$  is independent of  $W_{s_i} - W_{s_{i-1}}$ . Thus

$$\begin{aligned}
\mathbf{E}V_n &= \mathbf{E} \left[ \sum_{(a)} f(s_{i-1}, \omega)(W_{s_i} - W_{s_{i-1}})^2 - \sum_{(a)} f(s_{i-1}, \omega)(s_i - s_{i-1}) \right] \\
&= \sum_{(a)} \mathbf{E}f(s_{i-1}, \omega)(s_i - s_{i-1}) - \sum_{(a)} \mathbf{E}f(s_{i-1}, \omega)(s_i - s_{i-1}) \\
&= 0 .
\end{aligned}$$

Furthermore

$$\begin{aligned}
\mathbf{Var}V_n &= \mathbf{E} \left[ \sum_{(a)} f(s_{i-1}, \omega)(W_{s_i} - W_{s_{i-1}})^2 - \sum_{(a)} f(s_{i-1}, \omega)(s_i - s_{i-1}) \right]^2 \\
&= \mathbf{E} \sum_{(a)} f(s_{i-1}, \omega) f(s_{j-1}, \omega) [(W_{s_i} - W_{s_{i-1}})^2 - (s_i - s_{i-1})] \cdot [(W_{s_j} - W_{s_{j-1}})^2 - (s_j - s_{j-1})] \\
&= \sum_{(a)} \mathbf{E} f(s_{i-1}, \omega) f(s_{j-1}, \omega) [(W_{s_i} - W_{s_{i-1}})^2 - (s_i - s_{i-1})] \cdot [(W_{s_j} - W_{s_{j-1}})^2 - (s_j - s_{j-1})] .
\end{aligned}$$

If  $i < j$ , then  $f(s_{i-1}, \omega) f(s_{j-1}, \omega) [(W_{s_i} - W_{s_{i-1}})^2 - (s_i - s_{i-1})]$  is independent of  $[(W_{s_j} - W_{s_{j-1}})^2 - (s_j - s_{j-1})]$ , so we have  $\mathbf{E} f(s_{i-1}, \omega) f(s_{j-1}, \omega) [(W_{s_i} - W_{s_{i-1}})^2 - (s_i - s_{i-1})] \cdot [(W_{s_j} - W_{s_{j-1}})^2 - (s_j - s_{j-1})] = 0$ . Similarly, for  $i > j$  we also have  $\mathbf{E} f(s_{i-1}, \omega) f(s_{j-1}, \omega) [(W_{s_i} - W_{s_{i-1}})^2 - (s_i - s_{i-1})] \cdot [(W_{s_j} - W_{s_{j-1}})^2 - (s_j - s_{j-1})] = 0$ . This implies that

$$\begin{aligned}
\mathbf{Var}V_n &= \sum_{(a)} \mathbf{E} f^2(s_{i-1}, \omega) [(W_{s_i} - W_{s_{i-1}})^2 - (s_i - s_{i-1})]^2 \\
&= \sum_{(a)} \mathbf{E} f^2(s_{i-1}, \omega) \mathbf{E} [(W_{s_i} - W_{s_{i-1}})^2 - (s_i - s_{i-1})]^2 \\
&= \sum_{(a)} \mathbf{E} f^2(s_{i-1}, \omega) \mathbf{E} [(W_{s_i} - W_{s_{i-1}})^4 - 2(W_{s_i} - W_{s_{i-1}})^2(s_i - s_{i-1}) + (s_i - s_{i-1})^2] \\
&= \sum_{(a)} \mathbf{E} f^2(s_{i-1}, \omega) [3(s_i - s_{i-1})^2 - 2(s_i - s_{i-1})^2 + (s_i - s_{i-1})^2] \\
&= 2 \sum_{(a)} \mathbf{E} f^2(s_{i-1}, \omega) (s_i - s_{i-1})^2 \leq \frac{1}{2^{n-1}} \left( \max_{s \in [t_1, t_2]} \mathbf{E} f^2(s, \omega) \right) \sum_{(a)} (s_i - s_{i-1}) \rightarrow 0
\end{aligned}$$

as  $n \rightarrow \infty$ . This together with the fact that  $\mathbf{E}V_n = 0$  for any  $n$  implies the claim (14) of the Lemma.  $\square$

The argument above leads us to an Itô's formula for  $f(t, W_t)$ . Making use of the same methods, one can derive a more general Itô's formula for the solution  $X_t$  to the  $\Delta$ -stochastic differential equation (2). We will not repeat the proof, but we will claim the following Theorem.

**Theorem 2.** (General Itô's formula) *Let  $X_t$  be the solution to the  $\Delta$ -stochastic differential equation (2). For any function  $f \in \mathbf{C}^{(2)}(\mathbb{R}_+, \mathbb{R})$  and any  $t_1 \leq t_2$ ,  $t_1, t_2 \in [0, \infty)_{\mathbb{T}}$  we have*

$$\begin{aligned}
& f(t_2, X_{t_2}) - f(t_1, X_{t_1}) \\
= & \int_{t_1}^{t_2} b(s, X_s) \frac{\partial f}{\partial t}(s, X_s) \Delta s + \int_{t_1}^{t_2} \sigma(s, X_s) \frac{\partial f}{\partial x}(s, W_s) \Delta W_s + \frac{1}{2} \int_{t_1}^{t_2} \sigma^2(s, X_s) \frac{\partial^2 f}{\partial x^2}(s, W_s) \Delta s \\
& + \sum_{I_k \in \mathcal{I}, I_k \subset (t_1, t_2)} \left[ f(s_{I_k}^+, W_{s_{I_k}^+}) - f(s_{I_k}^-, W_{s_{I_k}^-}) - b(s_{I_k}^-, W_{s_{I_k}^-}) \frac{\partial f}{\partial t}(s_{I_k}^-, W_{s_{I_k}^-})(s_{I_k}^+ - s_{I_k}^-) \right. \\
& \quad \left. - \sigma(s_{I_k}^-, W_{s_{I_k}^-}) \frac{\partial f}{\partial x}(s_{I_k}^-, W_{s_{I_k}^-})(W_{s_{I_k}^+} - W_{s_{I_k}^-}) - \frac{1}{2} \sigma^2(s_{I_k}^-, W_{s_{I_k}^-}) \frac{\partial^2 f}{\partial x^2}(s_{I_k}^-, W_{s_{I_k}^-})(s_{I_k}^+ - s_{I_k}^-) \right] .
\end{aligned} \tag{15}$$

## 5 The stochastic exponential on time scales.

Our target in this section is to establish a closed-form formula for the *stochastic exponential* in the case of general time scales  $\mathbb{T}$ .

**Definition 2.** We say an adapted stochastic process  $A(t)$  defined on the filtered probability space  $(\Omega, \mathcal{F}_t, \mathbf{P})$  is stochastic regressive with respect to the Brownian motion  $W_t$  on the time scale  $\mathbb{T}$  if and only if for any right-scattered point  $t \in \mathbb{T}$  we have

$$(1 + A(t))(W_{\sigma(t)} - W_t) \neq 0, \quad \text{a.s. for all } t \in \mathbb{T}.$$

The set of stochastic regressive processes will be denoted by  $\mathcal{R}_W$ .

The following definition of a stochastic exponential was also introduced in [4].

**Definition 3.** (Stochastic Exponential) Let  $t_0 \in \mathbb{T}$  and  $A \in \mathcal{R}_W$ , then the unique solution of the  $\Delta$ -stochastic differential equation

$$\Delta X_t = A(t)X_t \Delta W_t, \quad X(t_0) = 1, \quad t \in \mathbb{T} \tag{16}$$

is called the stochastic exponential and is denoted by

$$X_\bullet = \mathcal{E}_A(\bullet, t_0).$$

We note that  $\mathcal{E}_A(t, t_0)$  as a solution to the equation (16) can be written into an integral equation

$$\mathcal{E}_A(t, t_0) = 1 + \int_{t_0}^t A(s) \mathcal{E}_A(s, t_0) \Delta W_s, \quad \text{for all } t \in \mathbb{T}. \tag{17}$$

We will be making use of the set-up we have in Section 4 about Itô's formula. Let  $t_0 < t$  and  $t_0, t \in \mathbb{T}$ . Let the sets  $C$  and  $\mathcal{I}$  be defined as in Section 4 corresponding to the interval  $[t_1, t_2] = [t_0, t]$ . Let  $I_k \in \mathcal{I}$  and  $I_k = (s_{I_k}^-, s_{I_k}^+)$ . We note that  $s_{I_k}^- = \rho(s_{I_k}^+)$ ,  $s_{I_k}^+ = \sigma(s_{I_k}^-)$ . Let

$$D(t, t_0) = \sum_{I_k \in \mathcal{I}, I_k \subset (t_0, t)} A(s_{I_k}^-)(W_{s_{I_k}^+} - W_{s_{I_k}^-}) - \frac{1}{2} \sum_{I_k \in \mathcal{I}, I_k \subset (t_0, t)} A^2(s_{I_k}^-)(s_{I_k}^+ - s_{I_k}^-) . \quad (18)$$

We define

$$U(t, t_0) = \prod_{I_k \in \mathcal{I}, I_k \subset (t_0, t)} \left[ 1 + A(s_{I_k}^-)(W_{s_{I_k}^+} - W_{s_{I_k}^-}) \right] , \quad (19)$$

$$V(t, t_0) = \exp \left( \int_{t_0}^t A(s) \Delta W_s - \frac{1}{2} \int_{t_0}^t A^2(s) \Delta s - D(t, t_0) \right) . \quad (20)$$

**Theorem 3.** (Stochastic Exponential on time scales) *The stochastic exponential has the closed-form expression*

$$\mathcal{E}_A(t, t_0) = U(t, t_0)V(t, t_0) .$$

**Proof.** Consider the process

$$Y_t = \int_{t_0}^t A(s) \Delta W_s - \frac{1}{2} \int_{t_0}^t A^2(s) \Delta s - D(t, t_0) .$$

Let us introduce another function  $\alpha(t)$  such that

$$\alpha(t) = \begin{cases} 0 & , \text{ when } t = s_{I_k}^- \text{ or } t = s_{I_k}^+ , \\ A(t) & , \text{ otherwise .} \end{cases}$$

We see now that the process  $Y_t$  is a solution to the  $\Delta$ -stochastic differential equation

$$\Delta Y_t = \alpha(t) \Delta W_s - \frac{1}{2} \alpha^2(t) \Delta s , \quad Y_{t_0} = 0 .$$

Notice that  $Y_{s_{I_k}^-} = Y_{s_{I_k}^+}$  for any  $I_k = (s_{I_k}^-, s_{I_k}^+) \in \mathcal{I}$ . Taking this into account, as well as the fact that  $\alpha(s) = 0$  whenever  $t = s_{I_k}^-$  or  $t = s_{I_k}^+$ , we can apply the general Itô's formula (15) to the function  $V(t, t_0) = \exp(Y_t)$  and we will get

$$\begin{aligned} & \exp(Y_t) - 1 \\ &= \int_{t_0}^t \alpha(s) \exp(Y_s) \Delta W_s - \frac{1}{2} \int_{t_0}^t \alpha^2(s) \exp(Y_s) \Delta s + \frac{1}{2} \int_{t_0}^t \alpha^2(s) \exp(Y_s) \Delta s \\ &= \int_{t_0}^t \alpha(s) \exp(Y_s) \Delta W_s . \end{aligned}$$

Thus

$$V(t, t_0) = 1 + \int_{t_0}^t \alpha(s) V(s, t_0) \Delta W_s ,$$

or in other words

$$\Delta V(t, t_0) = \alpha(t) V(t, t_0) \Delta W_t .$$

Let us now consider the function  $\mathcal{E}_A(t, t_0) = U(t, t_0) V(t, t_0)$ . We claim that

$$U(t, t_0) V(t, t_0) - 1 = \int_{t_0}^t A(s) U(s, t_0) V(s, t_0) \Delta W_s . \quad (21)$$

Notice that

$$U(s_{I_k}^+, t_0) = [1 + A(s_{I_k}^-)(W_{s_{I_k}^+} - W_{s_{I_k}^-})] U(s_{I_k}^-, t_0) = U(s_{I_k}^-, t_0) + A(s_{I_k}^-) U(s_{I_k}^-, t_0) (W_{s_{I_k}^+} - W_{s_{I_k}^-}) ,$$

i.e.,

$$U(s_{I_k}^+, t_0) - U(s_{I_k}^-, t_0) = A(s_{I_k}^-) U(s_{I_k}^-, t_0) (W_{s_{I_k}^+} - W_{s_{I_k}^-}) .$$

Using this fact, the above claimed identity (21) can be written as

$$\begin{aligned} & U(t, t_0) V(t, t_0) - 1 \\ &= \int_{t_0}^t \alpha(s) U(s, t_0) V(s, t_0) \Delta W_s + \sum_{I_k \in \mathcal{I}, I_k \subset (t_0, t)} A(s_{I_k}^-) U(s_{I_k}^-, t_0) V(s_{I_k}^-, t_0) (W_{s_{I_k}^+} - W_{s_{I_k}^-}) \\ &= \int_{t_0}^t \alpha(s) U(s, t_0) V(s, t_0) \Delta W_s + \sum_{I_k \in \mathcal{I}, I_k \subset (t_0, t)} V(s_{I_k}^-, t_0) (U(s_{I_k}^+, t_0) - U(s_{I_k}^-, t_0)) . \end{aligned} \quad (22)$$

In fact, with respect to the partition  $\pi^{(n)} : t_0 = s_0 < s_1 < \dots < s_{n-1} < s_n = t$  that we have been using, we have

$$\begin{aligned} & U(t, t_0) V(t, t_0) - 1 \\ &= \sum_{i=1}^n [U(s_i, t_0) V(s_i, t_0) - U(s_{i-1}, t_0) V(s_{i-1}, t_0)] \\ &= \sum_{i=1}^n [(U(s_i, t_0) - U(s_{i-1}, t_0))(V(s_i, t_0) - V(s_{i-1}, t_0)) + U(s_{i-1}, t_0)(V(s_i, t_0) - V(s_{i-1}, t_0)) \\ &\quad + V(s_{i-1}, t_0)(U(s_i, t_0) - U(s_{i-1}, t_0))] \\ &= \sum_{i=1}^n (U(s_i, t_0) - U(s_{i-1}, t_0))(V(s_i, t_0) - V(s_{i-1}, t_0)) + \sum_{i=1}^n U(s_{i-1}, t_0)(V(s_i, t_0) - V(s_{i-1}, t_0)) \\ &\quad + \sum_{i=1}^n V(s_{i-1}, t_0)(U(s_i, t_0) - U(s_{i-1}, t_0)) \\ &= (I) + (II) + (III) . \end{aligned}$$

Here



$$\begin{aligned}
(I) &= \sum_{i=1}^n (U(s_i, t_0) - U(s_{i-1}, t_0))(V(s_i, t_0) - V(s_{i-1}, t_0)) \quad , \\
(II) &= \sum_{i=1}^n U(s_{i-1}, t_0)(V(s_i, t_0) - V(s_{i-1}, t_0)) \quad , \\
(III) &= \sum_{i=1}^n V(s_{i-1}, t_0)(U(s_i, t_0) - U(s_{i-1}, t_0)) \quad .
\end{aligned}$$

We can apply the previous arguments and classify the intervals  $(s_{i-1}, s_i)$  into classes (a) and (b). Notice that on each interval  $(s_{I_k}^-, s_{I_k}^+)$ , the function  $V(t, t_0)$  remains constant and the function  $U(t, t_0)$  has a jump, and on each interval  $(s_{i-1}, s_i)$  in class (a) the function  $U(t, t_0)$  is a constant. This observation and similar arguments (which we leave to the reader) as in the previous section will enable us to prove that with probability 1, as  $n \rightarrow \infty$ , we will have

$$\begin{aligned}
(I) &\rightarrow 0 \quad , \\
(II) &\rightarrow \int_{t_0}^t \alpha(s) U(s, t_0) V(s, t_0) \Delta W_s \quad , \\
(III) &\rightarrow \sum_{I_k \in \mathcal{I}, I_k \subset (t_0, t)} V(s_{I_k}^-, t_0) (U(s_{I_k}^+, t_0) - U(s_{I_k}^-, t_0)) \quad .
\end{aligned}$$

So we proved (22) and thus (21).  $\square$

## 6 Change of measure and Girsanov's theorem on time scales.

We demonstrate in this section a change of measure formula (Girsanov's formula) for Brownian motion on time scales. Our analysis is based on the method of extension that was introduced in Section 3 (originally from [5]).

Let us consider two processes, the standard Brownian motion  $\{W_t\}_{t \in \mathbb{T}}$  on  $(\Omega, \mathcal{F}_t, \mathbf{P})$  on the time scale  $\mathbb{T}$ , and the process

$$B_t = W_t - \int_0^t A(s) \Delta s \quad ,$$

on the time scale  $t \in \mathbb{T}$ .

Let us consider an extension of the (probably random) function  $A(s)$  as in (1). Let us define the so obtained extension function to be  $\tilde{A}(s)$ . Recall that (1) implies that

$$\tilde{A}(s, \omega) = A(\sup[0, s]_{\mathbb{T}}, \omega) \quad .$$

Let  $\tilde{W}_t$  be a standard Brownian motion on  $[0, \infty)$ . If we define

$$\tilde{B}_t = \tilde{W}_t - \int_0^t \tilde{A}(s) ds \quad ,$$

then the process  $\tilde{B}_t$  agrees with  $B_t$  for any time point  $t \in \mathbb{T}$ .

For any  $t, t_0 \in \mathbb{T}$ ,  $t > t_0$ , let

$$\begin{aligned}\mathcal{G}_A(t, t_0) &= \exp \left( \int_0^t \tilde{A}(s) dW_s - \frac{1}{2} \int_0^t \tilde{A}^2(s) ds \right) \\ &= \exp \left( \int_0^t \tilde{A}(s) dW_s - \frac{1}{2} \int_0^t \tilde{A}^2(s) ds \right) \\ &= \exp \left( \int_0^t A(s) \Delta W_s - \frac{1}{2} \int_0^t A^2(s) \Delta s \right) .\end{aligned}\tag{23}$$

It is easy to see that the function  $\mathcal{G}_A(t, t_0)$  is the standard Girsanov's density function for the process  $\tilde{B}_t$  with respect to the standard Brownian motion  $\tilde{W}_t$ . Since  $\tilde{B}_t$  and  $\tilde{W}_t$  have the same distributions as  $B_t$  and  $W_t$  on the time scale  $\mathbb{T}$ , we conclude with the following two Theorems.

**Theorem 4.** (Novikov's condition on time scales) *If for every  $t \geq 0$  we have*

$$\mathbf{E} \exp \left( \int_0^t A^2(s) \Delta s \right) < \infty ,\tag{24}$$

*then for every  $t \geq 0$  we have*

$$\mathbf{E} \mathcal{G}_A(t, t_0) = 1 .$$

Let (24) be satisfied. Let  $T > 0$  and pick  $T > t_0, t_0, T \in \mathbb{T}$ . Consider a new measure  $\mathbf{P}^B$  on  $(\Omega, \mathcal{F}_t)$ , defined by its Radon–Nikodym derivative with respect to  $\mathbf{P}^W$ , as

$$\frac{d\mathbf{P}^B}{d\mathbf{P}^W} = \mathcal{G}_A(T, t_0) .$$

**Theorem 5.** (Girsanov's change of measure on time scales) *Under the measure  $\mathbf{P}^B$  the process  $B_t$ ,  $t \in [0, T]_{\mathbb{T}}$  is a standard Brownian motion on  $\mathbb{T}$ .*

## References

- [1] Ben Arous, G., Kumagai, T., Large deviations for Brownian motion on the Sierpinski gasket, *Stochastic Processes and their Applications*, **85**, 2000, pp. 225–235.
- [2] Bhamidi, S., Evans, S.N., Peled, R., Ralph, P., Brownian motion on time scales, basic hypergeometric functions, and some continued fractions of Ramanujan, *IMS Collections* 2008, Vol. 2, pp. 42–75.
- [3] Bohner, M., Peterson, A., *Dynamical Equations on Time Scales, An Introduction with Applications*, Birkhäuser Boston, Inc., Boston, MA, 2001.

- [4] Bohner, M., Sanyal, S., The stochastic dynamical exponential and geometric Brownian motion on isolated time scales, *Communications in Mathematical Analysis*, Vol. 8, No. 3, pp. 120–135 (2010).
- [5] Bohner, M., Stanzhytskyi, O.M., Bratochkina, A.O., Stochastic dynamical equations on general time scales, *Electronic Journal of Differential Equations*, 2013, No. 57, pp. 1–15.
- [6] Cont, R., Fournie, D.–A., Change of variable formulas for non–anticipative functionals on path space, *Journal of Functional Analysis*, **259**, pp. 1043–1072.
- [7] Du, Nguyen.H., Dieu, Nguyen.T., The first attempt on the stochastic calculus on time scale, *Stochastic Analysis and Applications*, **29**, pp. 1057–1080, 2011.
- [8] Grow, D., Sanyal, S., Brownian motion indexed by a time scale, *Stochastic Analysis and Applications*, 29, pp. 457–472, 2011.
- [9] Grow, D., Sanyal, S., The quadratic variation of Brownian motion on a time scale, *Statistics and Probability Letters*, **82**, 2012, pp. 1677–1680.
- [10] Grow, D., Sanyal, S., Existence and Uniqueness for Stochastic Dynamical Equations, *International Journal of Statistics and Probability*, Vol. 2, No. 2, 2013.
- [11] Kumagai, T., Short time asymptotic behavior and large deviations for Brownian motion on some affine nested fractals, *Publ. RIMS. Kyoto Univ.*, **33**, 1997, pp. 223–240.